Locally compact group extensions as a microcosm of the constructive Kasparov product

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Length functions and spectral triples

A *length function* on a discrete group G is a map $\ell : G \to [0, \infty)$ such that

- $\ell(gh) \leq \ell(g) + \ell(h)$,
- $\ell(g^{-1}) = \ell(g)$, and
- $\ell(e) = 0$

for $g, h \in G$ and e the unit. If $(1 + \ell^2)^{-1} \in C_0(G)$,

$$(C_r^*(G), \ell^2(G), M_\ell)$$

is a spectral triple, where M_ℓ is multiplication by ℓ [Con89]. This has been studied (and extended) by many authors, notably from the point of view of compact quantum metric spaces.

- Alas, because ℓ is positive, the class in $KK_*(C_r^*(G), \mathbb{C})$ is zero.
- Does this construction generalise to locally compact groups?

Weights on locally compact groups

Definition (M.)

Given a l.c. group G and a finite-dimensional complex vector space V, a *weight* is a continuous function

$$\ell : G \to \operatorname{End} V.$$

We require that $\ell^* = \ell$ and, if V is $\mathbb{Z}/2\mathbb{Z}$-graded, that ℓ be odd. We say that ℓ is

- *proper* if $(1 + \ell^2)^{-1} \in C_0(G, \operatorname{End} V) = C_0(G) \otimes \operatorname{End} V$; and
- *translation-bounded* if, for all $g \in G$,

  $$\sup_{h \in G} \|\ell(gh) - \ell(h)\| < \infty$$

  and there exists a neighbourhood U of the identity in G such that

  $$\sup_{g \in U, h \in G} \|\ell(gh) - \ell(h)\| < \infty.$$

Fissured Fell bundles over locally compact groups

Definition (M.)

Let $\mathscr{B}$ be a Fell bundle over a l.c. group G. Let $\hat{\mathscr{B}}$ be the continuous field of C*-algebras over G with fibre

$$\hat{B}_g = B_g \otimes_{B_e} B_{g^{-1}} = \mathrm{End}^0_{B_e}(B_g) \trianglelefteq B_e$$

at $g \in G$. The C*-algebra of its continuous sections $\Gamma_0(\hat{\mathscr{B}})$ is an ideal of $C_0(G, B_e)$. We say that $\mathscr{B}$ is *fissured* if $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$.

If $\Gamma_0(\hat{\mathscr{B}}) = C_0(G, B_e)$, i.e. $\hat{B}_g = B_e$ for all $g \in G$, $\mathscr{B}$ is *saturated*.

Fissuration is a generalisation of the spectral subspace assumption of [CNNR11], which is the case $G = \mathbb{Z}$. An example of a Fell bundle which is fissured but not saturated is that associated to the partial Bernoulli action of a discrete group G.

Two unbounded Kasparov modules

Theorem (M.)

Let G be a l.c. group, V a finite-dimensional complex vector space, and $\ell : G \to \operatorname{End} V$ a proper t.-b. weight.

- *If $\mathscr{B}$ is a fissured Fell bundle over G,*

  $$(C_r^*(\mathscr{B}), (L^2(\mathscr{B}) \otimes V)_{B_e}, M_\ell)$$

  is a $\hat{G}$-equivariant unbounded Kasparov $C_r^*(\mathscr{B})$-B_e-module.
- *If A is a G-C^*-algebra,*

  $$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

  is a G-equivariant unbounded Kasparov A-$C_0(G, A)$-module.

One can think of this as a wide-ranging generalisation of the Pimsner–Voiculescu extension class.

Back to the easy case

For $\mathscr{B}$ the group bundle, we obtain a spectral triple for $C_r^*(G)$.

Corollary

If G is a l.c. group and $\ell : G \to \operatorname{End} V$ is a proper t.-b. weight,

$$\big(C_r^*(G), L^2(G, V), M_\ell\big)$$

is a $\hat{G}$-equivariant spectral triple and

$$\Big(\mathbb{C}, C_0(G, V)_{C_0(G)}, \ell\Big)$$

is a G-equivariant unbounded Kasparov $\mathbb{C}$-$C_0(G)$-module, related by Baaj–Skandalis duality.

Nontriviality in KK-theory

$$
\begin{tikzcd}
KK_*^G(\mathbb{C}, C_0(G)) \arrow[d, "J^G"', "\sim"] \arrow[dr, "j_r^G"] & \\
KK_*^{\hat{G}}(C_r^*(G), C_0(G) \rtimes_r G) \arrow[r, "r^{\hat{G},1}"'] \arrow[d, "\otimes[L^2(G)]"', "\sim"] & KK_*(C_r^*(G), C_0(G) \rtimes_r G) \arrow[d, "\otimes[L^2(G)]"', "\sim"] \\
KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) \arrow[r, "r^{\hat{G},1}"'] & KK_*(C_r^*(G), \mathbb{C})
\end{tikzcd}
$$

Weights as directed length functions

Consider a compactly generated abelian group $G = \mathbb{R}^m \times \mathbb{Z}^n \times K$.
Define $\ell : G \to \mathscr{C}\ell_{m+n}$ by

$$\ell(x_1, \dots, x_m, k_1, \dots, k_n) = x_1\gamma_1 + \cdots x_m\gamma_m + k_1\gamma_{m+1} + \cdots + k_n\gamma_{m+n}.$$

For groups acting on many CAT(0) spaces, such as

- Hadamard manifolds,
- (the geometric realisations of) trees and
- Euclidean and hyperbolic Bruhat–Tits buildings,

there exist natural weights given by directed length functions. One can also show nontriviality in KK-theory in these cases.

Weights as directed length functions

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define $\ell_a, \ell_b : F_n \to \mathbb{C}$ be given on a reduced word w of length $|w|$ by

$$\ell_a(w) = \begin{cases} +|w| & w \text{ ends in } a \\ -|w| & w \text{ ends in } a^{-1}, b, \text{ or } b^{-1} \end{cases}$$
$$\ell_b(w) = \begin{cases} +|w| & w \text{ ends in } b \\ -|w| & w \text{ ends in } a, a^{-1}, \text{ or } b^{-1} \end{cases}.$$

These are proper and t.-b. and give rise to distinct (nontrivial) KK-classes.

Let G be a real reductive Lie group and K its maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition; we have $G = K \exp \mathfrak{p}$. The Killing form B on $\mathfrak{g}$ is an inner product on $\mathfrak{p}$. Define $\ell : G \to \mathscr{C}\ell(\mathfrak{p}, B)$ by $\ell(k \exp(X)) = X$.

Building new weights from old

- A proper t.-b. weight ℓ_G on a l.c. group G restricts to one on a cocompact closed subgroup H of G.
- A proper t.-b. weight ℓ_H on a cocompact closed subgroup H of a l.c. group G can be induced to one on G. Let

$$\widehat{\ell_H} : g \mapsto \int_H c(gs)^2 \ell_H(s^{-1})\, d\mu_H(s)$$

  where $c \in C_b(G)$ is a cut-off function for the right action of H on G.
- Proper t.-b. weights $\ell_H : H \to V_H, \ell_N : N \to V_N$ can be combined into

$$\ell_{H\times N} = \ell_H \,\tilde{\otimes}\, 1 + 1 \,\tilde{\otimes}\, \ell_N : G \to V_H \,\tilde{\otimes}\, V_N .$$

Group extensions

Let

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0$$

be an exact sequence of l.c. groups. Given proper t.-b. weights ℓ_N and ℓ_H on N and H, we would like to build ℓ_G. This amounts to the **constructive unbounded Kasparov product** and also requires **reduced partial-cross sectional Fell bundles**.

- If $\mathscr{B}$ is a fissured Fell bundle over G, with $\mathscr{C}$ the reduced partial-cross sectional Fell bundle over $H = G/N$,

$$(C_r^*(\mathscr{B}) \cong C_r^*(\mathscr{C}), (L^2(\mathscr{C}) \otimes V_H)_{C_N}, M_{\ell_H})$$

  is a $\hat{G}$-equivariant unbounded Kasparov module.

- If A is a G-C*-algebra,

$$(C_0(H, A), C_0(G, A \otimes V_N)_{C_0(G,A)}, \widehat{\ell_N})$$

  is a G-equivariant unbounded Kasparov module.

The constructive Kasparov product applied

Translating the conditions of [LM19], we obtain

Theorem (M.)

Let

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0$$

be an exact sequence of l.c. groups. Suppose that we have proper t.-b. weights $\ell_N : N \to \operatorname{End} V_N$ *and* $\ell_H : H \to \operatorname{End} V_H$*. Suppose that* $\widetilde{\ell_N} : G \to V_N$ *is a t.-b. weight such that*

$$\sup_{n \in N} \|\widetilde{\ell_N}(\iota(n)) - \ell_N(n)\| < \infty.$$

Then $\ell_G := \pi^*(\ell_H) \,\tilde{\otimes}\, 1 + 1 \,\tilde{\otimes}\, \widetilde{\ell_N} : G \to V_G$ *is a proper t.-b. weight.*

The constructive Kasparov product applied

Theorem (cont.)

Further, for $\mathscr{B}$ a fissured Fell bundle over G,

$$^{\hat{G}}(C_r^*(\mathscr{B}), (L^2(\mathscr{B}) \otimes V_G)_{B_e}, M_{\ell_G})$$

represents the Kasparov product

$$^{\hat{G}}(C_r^*(\mathscr{B}), (L^2(\mathscr{C}) \otimes V_H)_{C_N}, M_{\ell_H}) \\ \otimes_{C_r^*(\mathscr{B}_N)} {}^{\hat{G}}(C_r^*(\mathscr{B}_N), (L^2(\mathscr{B}_N) \otimes V_N)_{B_e}, M_{\ell_N})$$

and, for A a G-C-algebra,*

$$^{G}(A, C_0(G, A \otimes V_G)_{C_0(G,A)}, \ell_G)$$

represents the Kasparov product

$$^{G}(A, C_0(H, A \otimes V_H)_{C_0(H,A)}, \ell_H) \\ \otimes_{C_0(H,A)} {}^{G}(C_0(H, A), C_0(G, A \otimes V_N)_{C_0(G,A)}, \widehat{\ell_N}).$$

The constructive Kasparov product applied

In practice, the only difficult point is the translation-boundedness of $\widetilde{\ell_N}$. Indeed, the weight $\widehat{\ell_N}$ automatically satisfies the other condition.

Example: the universal cover of $SL(2,\mathbb{R})$

The universal cover of $SL(2,\mathbb{R})$ fits into the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{SL}(2,\mathbb{R}) \longrightarrow SL(2,\mathbb{R}) \longrightarrow 0 \ .$$

As this is a central extension, we may represent elements of $\widetilde{SL}(2,\mathbb{R})$ as pairs in $SL(2,\mathbb{R}) \times \mathbb{Z}$, with

$$(g_1, n_1)(g_2, n_2) = (g_1 g_2, n_1 + n_2 + W(g_1, g_2))$$

for some 2-cocycle $W : \widetilde{SL}(2,\mathbb{R}) \times \widetilde{SL}(2,\mathbb{R}) \to \{0, \pm 1\}$.

Let $\ell_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ take $n \mapsto n$. Can't take $\widetilde{\ell_{\mathbb{Z}}} : (g, n) \mapsto n$ as it's not continuous. Instead, Iwasawa-decompose

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $x + iy$ is in the upper-half plane and $\theta \in [-\pi, \pi)$. Let

$$\widetilde{\ell_{\mathbb{Z}}}(g, n) = n + \frac{\theta}{2\pi}.$$

Example: the universal cover of $SL(2,\mathbb{R})$

First,

$$\sup_{n\in\mathbb{Z}} \|\widetilde{\ell_{\mathbb{Z}}}(e,n) - \ell_{\mathbb{Z}}(n)\| = 0.$$

Second, by [Asa70], with $\arg : \mathbb{C} \setminus \{0\} \to [-\pi,\pi)$,

$$\widetilde{\ell_{\mathbb{Z}}}((g_1,n_1)(g_2,n_2)) - \widetilde{\ell_{\mathbb{Z}}}(g_2,n_2) = n_1 + \frac{1}{2\pi}\arg((x_2+iy_2)\sin\theta_1 + \cos\theta_1),$$

which is bounded by $|n_1| + \frac{1}{2}$. (This latter is just a consequence of the boundedness of the cocycle W.) For the open neighbourhood

$$U = \{(g,0) \mid \theta \in (-\pi,\pi)\}$$

of the identity $(e,0) \in \widetilde{SL}(2,\mathbb{R})$,

$$\sup_{(g_1,n_1)\in U, (g_2,n_2)\in\widetilde{SL}(2,\mathbb{R})} \left\|\widetilde{\ell_{\mathbb{Z}}}((g_1,n_1)(g_2,n_2)) - \widetilde{\ell_{\mathbb{Z}}}(g_2,n_2)\right\| \leq \frac{1}{2}.$$

Hence $\widetilde{\ell_{\mathbb{Z}}}$ is translation-bounded and we can apply the Theorem.

Semidirect products

Let $\varphi : H \to \mathrm{Aut}(N)$ and $G = H \ltimes_\varphi N$ be a semidirect product of l.c. groups. For $s, t \in H$ and $m, n \in N$,

$$(s, m)(t, n) = (st, \varphi_{t^{-1}}(m)n).$$

Suppose we have a proper t.-b. weight $\ell_N : N \to \mathrm{End}\, V_N$. We need a t.-b. weight $\widetilde{\ell_N} : G \to V_N$ such that

$$\sup_{m \in N} \|\widetilde{\ell_N}(e, m) - \ell_N(m)\| < \infty.$$

If we naïvely define

$$\widetilde{\ell_N}(s, m) = \ell_N(m)$$

then

$$\widetilde{\ell_N}((s, m)(t, n)) - \widetilde{\ell_N}(t, n) = \ell_N(\varphi_{t^{-1}}(m)n) - \ell_N(n).$$

!!! This is usually not uniformly bounded in $t \in H$ and $n \in N$.

Failure of the constructive unbounded Kasparov product

If $\varphi : H \to \mathrm{Aut}(N)$ is

- ‘elliptic/isometric/compact’, everything works fine.
- ‘parabolic/polynomially unbounded/nilpotent’, one can make a tangled spectral triple [FGM26]. This involves the use of a fractional power functional calculus and also works for nilpotent G.
- ‘hyperbolic/exponentially unbounded/generic’, both these approaches fail.

Lion taming by logarithm

The use of logarithmic dampening in NCG is due to [GMR19].

Lemma

Let E be a normed real vector space. Let $L : E \to E$ be given by

$$L(x) = \frac{\log(1 + \|x\|)}{\|x\|} x.$$

For $x, y \in E$ and $\alpha > 0$,

$$\|L(x + \alpha y) - L(y)\| \leq |\log \alpha| + 2 \log(1 + \tfrac{1}{2}\|x\|).$$

So the bound is independent of y and continuous in x and α.
(N.B. L is not a linear map.)

Lion taming by logarithm

Now work with $L \circ \ell_N : N \to \operatorname{End} V_N$, which is still proper and t.-b.. $L \circ \ell_N$ **preserves all the KK-theoretic content** of ℓ_N.

Theorem (M.)

Let $G = H \ltimes_\varphi N$ be a semidirect product of l.c. groups. Let $\ell_N : N \to \operatorname{End} V_N$ be a proper t.-b. weight. Suppose that N has a compact generating set S and let

$$j_S(t) = \sup_{m \in S, n \in N} \left\| \ell_N(\varphi_{t^{-1}}(m)n) - \ell_N(n) \right\|.$$

Then

$$\widetilde{L \circ \ell_N}(t, n) = L\big(j_S(t)^{-1} \ell_N(n)\big).$$

gives a t.-b. weight and

$$\sup_{n \in N} \| \widetilde{L \circ \ell_N}(e, n) - L \circ \ell_N(n) \| < \infty.$$

⊕ The dual Dirac element for almost connected groups

Even when the extension is not split (and G is not a semidirect product) a similar technique can often be used. As an application of this, we can produce an explicit representative for the dual Dirac element for any almost connected group G.

If K is a maximal compact subgroup of G then $X := G/K$ is diffeomorphic to $\mathbb{R}^n$. The dual Dirac element is the unique $\beta \in KK^G(\mathbb{C}, \Gamma_0(\mathscr{C}\ell(T^*X)))$ such that $\alpha \otimes_{\mathbb{C}} \beta = 1 \in KK^G(\Gamma_0(\mathscr{C}\ell(T^*X)), \Gamma_0(\mathscr{C}\ell(T^*X)))$, where α is the Dirac element. Then $\gamma := \beta \otimes_{\Gamma_0(\mathscr{C}\ell(T^*X))} \alpha$.

In [Kas88], Kasparov shows that β exists by an inductive argument, using the Technical Theorem.

⌖ The dual Dirac element for almost connected groups

Lemma

G has a series of normal subgroups

$$\{e\} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_m \subset G$$

such that each N_{k+1}/N_k is either compact or isomorphic to $\mathbb{R}^{n_k}$ or $\mathbb{Z}^{n_k}$, and G/N_m is a centreless semisimple Lie group.

Let K' a maximal compact subgroup of G/N_m. Then $(G/N_m)/K'$ is CAT(0) and diffeomorphic to $\mathbb{R}^{n-\sum_k n_k}$, meaning we can build a directed length function ℓ_{G/N_m}. Successively using logarithmic dampening and the constructive Kasparov product, we arrive at a proper t.-b. weight $\ell_G : G \to \mathscr{C}\ell_n$.

The dual Dirac element for almost connected groups

To see the relationship between this and β, define $\omega : \Gamma_0(\mathscr{C}\ell(T^*X)) \to C_0(G, \mathscr{C}\ell(T^*_{eK}X))$ by

$$\omega(\sigma)(g) = g^{-1} \cdot \sigma(gK).$$

We tacitly identify $\mathscr{C}\ell(T^*_{eK}X)$ with $\mathscr{C}\ell_n$.

Now β is represented by

$$^G(\mathbb{C}, \Gamma_0(\mathscr{C}\ell(T^*X))_{\Gamma_0(\mathscr{C}\ell(T^*X))}, \vartheta)$$

where one defines ϑ so that

$$\omega(\vartheta) = \ell.$$

(This will make sense because $\ell(gk) = \ell(g)$ for all $g \in G$ and $k \in K$.)

Thanks

- Tetsuya Asai, *The reciprocity of Dedekind sums and the factor set for the universal covering group of* $SL(2,\mathbb{R})$, Nagoya Mathematical Journal **37** (1970), 67–80.
- Alan L. Carey, Sergey Neshveyev, Ryszard Nest, and Adam Rennie, *Twisted cyclic theory, equivariant KK-theory and KMS states*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2011** (2011), no. 650, 161–191.
- Alain Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory and Dynamical Systems **9** (1989), no. 2, 207–220.
- Magnus Fries, Magnus Goffeng, and Ada Masters, *Parabolic noncommutative geometry*, Advances in Mathematics **499** (2026), 111057.
- Magnus Goffeng, Bram Mesland, and Adam Rennie, *Untwisting twisted spectral triples*, International Journal of Mathematics **30** (2019), no. 14, 1950076.
- G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Inventiones Mathematicae **91** (1988), no. 1, 147–201.
- Matthias Lesch and Bram Mesland, *Sums of regular self-adjoint operators in Hilbert-C*-modules*, Journal of Mathematical Analysis and Applications **472** (2019), no. 1, 947–980.