Spectral triples for group C*-algebras with nontrivial index theory

Ada Masters

Centre for Mathematical Sciences
Lund University

Groups, Operator Algebras and Quantum Day
University of Copenhagen

20 March

All results in this talk can be found in my PhD thesis, which is available on my website, `ada-masters.com`, and will appear in joint work with Adam Rennie and Anne Thomas.

Length functions and spectral triples à la Connes

A *length function* on a discrete group G is a map $\ell : G \to [0, \infty)$ such that

1. $\ell(gh) \leq \ell(g) + \ell(h)$,
2. $\ell(g^{-1}) = \ell(g)$, and
3. $\ell(e) = 0$

for $g, h \in G$ and e the identity. If, in addition, $(1 + \ell^2)^{-1} \in C_0(G)$, then

$$(C_r^*(G), \ell^2(G), M_\ell)$$

is a spectral triple, where the operator M_ℓ is multiplication by ℓ [Con89]. This has been studied (and extended) by many authors, notably from the point of view of compact quantum metric spaces. Alas, because ℓ is positive, the class in $KK_*(C_r^*(G), \mathbb{C})$ is zero. Also, does this construction generalise to locally compact groups?

Weights on locally compact groups

Definition

Given a locally compact group G and a finite-dimensional complex vector space V, a *weight* is a continuous function

$$\ell : G \to \operatorname{End} V.$$

If V is $\mathbb{Z}/2\mathbb{Z}$-graded, we require that ℓ be odd. We say that ℓ is

- self-adjoint if $\ell^* = \ell$;
- *proper* if $(1 + \ell^*\ell)^{-1} \in C_0(G, \operatorname{End} V) = C_0(G) \otimes \operatorname{End} V$; and
- *translation-bounded* if, for all $g \in G$, $\sup_{h\in G} \|\ell(gh) - \ell(h)\| < \infty$ and there exists a neighbourhood U of the identity in G such that $\sup_{g\in U, h\in G} \|\ell(gh) - \ell(h)\| < \infty$.

Fissured Fell bundles over locally compact groups

Definition

Let $\mathscr{B}$ be a Fell bundle over a locally compact group G. Let $\hat{\mathscr{B}}$ be the continuous field of C*-algebras over G with fibre

$$\hat{B}_g = B_g \otimes_{B_e} B_{g^{-1}} = \mathrm{End}^0_{B_e}(B_g) \trianglelefteq B_e$$

at $g \in G$. The C*-algebra of its continuous sections $\Gamma_0(\hat{\mathscr{B}})$ is an ideal of $C_0(G, B_e)$. We say that $\mathscr{B}$ is *fissured* if $\Gamma_0(\hat{\mathscr{B}})$ is a complemented ideal of $C_0(G, B_e)$. If $\Gamma_0(\hat{\mathscr{B}}) = C_0(G, B_e)$, i.e. $\hat{B}_g = B_e$ for all $g \in G$, $\mathscr{B}$ is *saturated*.

Fissuration is a generalisation of the spectral subspace assumption of [CNNR11], which is the case $G = \mathbb{Z}$. An example of a Fell bundle which is fissured but not saturated is that associated to the partial Bernoulli action of a discrete group G.

Two unbounded Kasparov modules

Theorem

Let G be a locally compact group, V a finite-dimensional complex vector space, and $\ell : G \to \operatorname{End} V$ a self-adjoint, proper, translation-bounded weight.
Let $\mathscr{B}$ be a Fell bundle over G. If $\mathscr{B}$ is fissured,

$$(C_r^*(\mathscr{B}), L^2(\mathscr{B}) \otimes V, M_\ell)$$

is an isometrically $\hat{G}$-equivariant unbounded Kasparov $C_r^*(\mathscr{B})$-B_e-module.
Let A be a G-C^*-algebra. Then

$$(A, C_0(G, A \otimes V)_{C_0(G,A)}, \ell)$$

is a uniformly G-equivariant unbounded Kasparov A-$C_0(G,A)$-module.

One can think of this as a wide-ranging generalisation of the Pimsner–Voiculescu extension class.

Back to the easy case

In particular, for $\mathscr{B}$ the group bundle, we obtain a spectral triple for $C_r^*(G)$.

Corollary

Let G be a locally compact group and V be a finite-dimensional complex vector space. Let $\ell : G \to \operatorname{End} V$ be a self-adjoint, proper, and translation-bounded weight. Then

$$\left(C_r^*(G), L^2(G, V), M_\ell\right)$$

is an isometrically $\hat{G}$-equivariant spectral triple and

$$\left(\mathbb{C}, C_0(G, V)_{C_0(G)}, \ell\right)$$

is a G-equivariant unbounded Kasparov $\mathbb{C}$-$C_0(G)$-module, related by Baaj–Skandalis duality.

Nontriviality in KK-theory

$$
\begin{array}{ccc}
KK_*^{G}(\mathbb{C}, C_0(G)) & & \\
{\scriptstyle J^G}\big\downarrow{\scriptstyle \sim} & \overset{j_r^G}{\searrow} & \\
KK_*^{\hat{G}}(C_r^*(G), C_0(G) \rtimes_r G) & \xrightarrow[r^{\hat{G},1}]{} & KK_*(C_r^*(G), C_0(G) \rtimes_r G) \\
{\scriptstyle \otimes[L^2(G)]}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \otimes[L^2(G)]}\big\downarrow{\scriptstyle \sim} \\
KK_*^{\hat{G}}(C_r^*(G), \mathbb{C}) & \xrightarrow[r^{\hat{G},1}]{} & KK_*(C_r^*(G), \mathbb{C})
\end{array}
$$

Building new weights from old

- Any s.a. proper t.b. weight ℓ_G on a locally compact group G can be restricted to one on any closed subgroup H of G.
- Any s.a. proper t.b. weight ℓ_H on a cocompact subgroup H of a locally compact group G can be induced to one on G.
- Let
$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 0$$
be an exact sequence of locally compact groups. Given s.a. proper t.b. weights ℓ_N and ℓ_H on N and H, **satisfying certain conditions**, we can build a weight for G.

This latter is an instance of the unbounded Kasparov product and requires the group extension Fell bundle. An example for which it succeeds immediately is the universal cover of $SL(2,\mathbb{R})$. For any semidirect product $\mathbb{R}^n \rtimes \mathbb{R}$, we can also proceed, with the aid of logarithmic dampening. For nilpotent groups, this line of thinking leads to tangled spectral triples, see joint work with Magnus Fries and Magnus Goffeng [FGM25].

CAT(0) spaces

A *geodesic space* is a metric space (X, d) in which every two points are joined by a geodesic. In a geodesic space, there is a notion of *Alexandrov angle* $\angle(c, c')$ between two geodesics c and c' in X with $c(0) = c'(0)$. The *space of directions* $S_x(X)$ at a point $x \in X$ is the set of geodesics emanating from x, modulo the equivalence relation of zero Alexandrov angle.

A geodesic space (X, d) is *CAT(0)* if for every geodesic triangle (c, c', c'') in X with distinct vertices, a triangle in $\mathbb{R}^2$ with two side lengths $d(c)$, $d(c')$ and interior angle $\angle(c, c')$ has its third side no longer than $d(c'')$.

The CAT(0) condition should be thought of as a non-positive curvature condition. Examples of CAT(0) spaces include

- simply connected complete Riemannian manifolds with everywhere non-positive sectional curvature (Hadamard manifolds);
- (the geometric realisations of) trees; and
- Euclidean and hyperbolic Bruhat–Tits buildings.

Directed length functions

For points x and y of a geodesic metric space X, we denote by $\upsilon(x, y) \in S_y(X)$ the direction of the geodesic from x to y as it reaches y, where $S_y(X)$ is the space of directions.

Theorem

Let G be a locally compact group acting isometrically on a CAT(0) space (X, d). Suppose that at a point $x_0 \in X$, the space of directions $S_{x_0}(X)$ is isometric to a sphere $\mathbf{S}^{n-1} \subseteq \mathbb{R}^n$. Let V be a Clifford module for the Clifford algebra $\mathscr{C}\ell_n$. Define the function $\ell : G \to \operatorname{End} V$ by

$$\ell(g) = d(g^{-1} \cdot x_0, x_0)\upsilon(g^{-1} \cdot x_0, x_0)$$

where $\upsilon(g^{-1} \cdot x_0, x_0) \in S_{x_0}(X) \cong \mathbf{S}^{n-1} \subseteq \mathbb{R}^n \subseteq \mathscr{C}\ell_n$ acts by Clifford multiplication on V. Then ℓ is self-adjoint and translation bounded. If G acts properly on X, ℓ is proper.

Nontriviality in KK-theory

We also prove a nontriviality result for the resulting KK-classes by pairing them with a suitable 'Dirac class'.

Theorem

Let G be a locally compact group acting properly and isometrically on a CAT(0) space (X, d). Let A be a G-C^*-algebra. Suppose that there is a complete subspace Y of X such that

- *every path component of Y is a convex subset of X (Y may have infinitely many path components.);*
- *Y is isometric to a spin^c Riemannian n-manifold; and*
- *Y contains a neighbourhood of a point $x_0 \in X$.*

Let $x_1 \in X$ be a point not in Y but with $S_{x_1}(X)$ isometric to a sphere $\mathbf{S}^{m-1} \subseteq \mathbb{R}^m$. Let V_0 and V_1 be Clifford modules for $\mathscr{C}\ell_n$ and $\mathscr{C}\ell_m$ respectively, with V_0 irreducible. Define the weights

$$\ell_0 : G \to \operatorname{End} V_0 \qquad\qquad \ell_1 : G \to \operatorname{End} V$$
$$g \mapsto d(g^{-1}x_0, x_0)\upsilon(g^{-1}x_0, x_0) \qquad\qquad g \mapsto d(g^{-1} \cdot x_1, x_1)\upsilon(g^{-1} \cdot x_1, x_1),$$

representing classes $\sigma_A([\ell_0]), \sigma_A([\ell_1]) \in KK^G_*(A, C_0(G, A))$ and $[M_{\ell_0}], [M_{\ell_1}] \in KK^{\hat{G}}_*(A \rtimes_r G, A)$.
For any closed subgroup H of G preserving Y, let $\eta_H : C_0(Y, A)^H \to A$ be the $*$-homomorphism given by evaluating at x_0, giving a class $[\eta_H] \in KK_0(C_0(Y, A)^H, A)$. For $A = \mathbb{C}$, $[\eta_H] \in KK_0(C_0(Y/H), \mathbb{C})$ is nonzero if and only if H acts cocompactly on Y.
If there exists a closed subgroup H of G such that H preserves Y and acts by pin^c automorphisms and $[\eta_H]$ is nonzero then $\sigma_A([\ell_0]) \in KK^G_n(\mathbb{C}, C_0(G))$ is nonzero and not equal to $\sigma_A([\ell_1])$.
If G itself preserves Y, acts by spin^c automorphisms, and $[\eta_G]$ is nonzero then $r^{\hat{G},1}([M_{\ell_0}]) \in KK_n(A \rtimes_r G, A)$ is nonzero and not equal to $r^{\hat{G},1}([M_{\ell_1}])$.

Idea: Let the free group F_2 act on the tree and build directed length functions ℓ_0, ℓ_1, ℓ_2 using x_0, x_1, x_2 as basepoints.

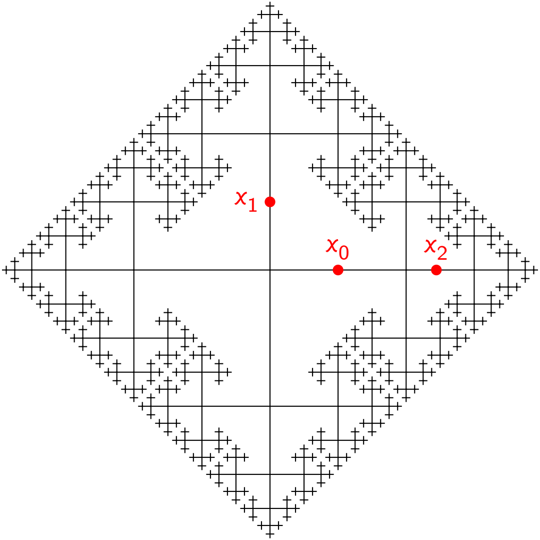
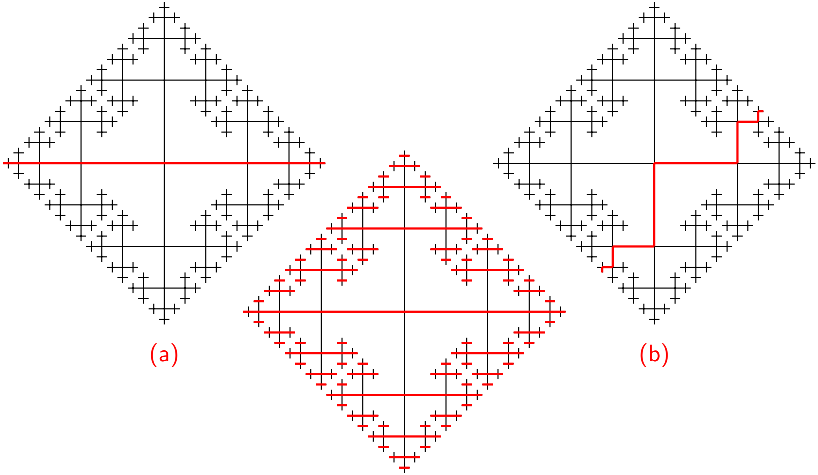


Figure: The Cayley graph of F_2.

Idea: find submanifolds (with possibly infinitely many connected components) preserved by the group or by subgroups to show that KK-classes given by ℓ_0, ℓ_1, ℓ_2 are nontrivial and/or distinct.



(a) (b) (c)

Thanks

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